THE EQUATIONS OF TURBULENT FLOW FOR AN INCOMPRESSIBLE FLUID

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The Turbulent flow of an incompressible fluid basically satisfies the familiar Reynolds equations (see, for example, [1]). However, because of their extreme complexity, it has not been possible to solve these equations. We are therefore confronted with the problem of deriving other equations which are simpler, but which retain all the fundamental features of turbulent flow, and which—to a greater extent than the Reynolds equations—lend themselves to investigation and approximate solution.

Below we present the derivation of such equations. The derivation is based on the assumption that the wavelength λ of the turbulent pulsations is significantly smaller than the dimension L of the system. This assumption permits us to write a system of ordinary first-order differential equations for the Fourier amplitude of the rapidly changing component of a velocity field, and then—after introduction of the distribution function—to derive an equation for the distribution function.

The equation for an averaged velocity field does not essentially differ from the corresponding Reynolds equation.

The derived system of equations can be used for a numerical calculation of both the spectrum of small-scale pulsations for a specified average field and the average field itself, the latter formed as a result of these pulsations.

1. Pulsations with a small wavelength. Let the flow of the fluid be described by the Reynolds number R. Then, for the wave numbers of these pulsations (perturbations) which are rapidly attenuated as a result of viscosity, we find the following inequality to be valid: $k = 2\pi/\lambda \geq$ $\geq R/L$, where L denotes the dimensions of the system. Viscosity no longer exerts a significant effect on perturbations with wave numbers smaller than R/L, but these are attenuated because of the stability of the laminar flow, so long as $R < R_*$, where R_* is the critical Reynolds number. Generally speaking, the nonviscous mechanism of attenuation is effective in the case of perturbations whose wave numbers do not exceed R_*/L.

As soon as R exceeds R_* we find nonattenuating perturbations with wave numbers in the interval $R/L > k > R_*/L$.

However, since it is usual that $R_* \approx 1000$, it follows that the perturbations exhibiting the smallest wavelength relative to the dimensions of the system are the ones responsible for the onset of turbulence.

Let us assume that perturbations of such large wave numbers predominate not only at the initial stage of the turbulization, but also in a system with developed turbulence. In this case, the velocity $\mathbf{u}(\mathbf{r}, t)$ and the pressure $p(\mathbf{r}, t)$ of the turbulent flow can be presented in the form of the sum of the two terms

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{U}(\mathbf{r}, t) + \mathbf{u}'(\mathbf{r}, t), \quad p(\mathbf{r}, t) = P(\mathbf{r}, t) + p'(\mathbf{r}, t). \quad (1.1)$$

The functions $U(\mathbf{r}, t)$ and $P(\mathbf{r}, t)$ depend smoothly on the coordinates and on time, varying significantly only at distances commensurate with the characteristic dimensions of the system, whereas $u'(\mathbf{r}, t)$ and $p'(\mathbf{r}, t)$ oscillate rapidly in space and time.

Let us introduce the Lagrange variables associated with the velocity field:

$$\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t), \quad \mathbf{U}(\mathbf{r}_0, t) = (\partial \mathbf{r} / \partial t)_{\mathbf{r}_0}.$$

Let us seek $u'(r_0, t)$ and $p'(r_0, t)$ in the form

$$\mathbf{u}'(\mathbf{r}_0, t) = \sum_{\mathbf{k}} \mathbf{u}(\mathbf{k}, \mathbf{r}_0, t) e^{i\mathbf{k}\mathbf{r}_0}, \quad p'(\mathbf{r}_0, t) = \sum_{k} p(\mathbf{k}, \mathbf{r}_0, t) e^{i\mathbf{k}\mathbf{r}_0}. \quad (1.2)$$

The amplitudes $u(\mathbf{k}, r_0, t)$ and $p(\mathbf{k}, r_0, t)$ contain only a smooth dependence on r_0, with the strong dependence contained in the exponential factors.

The volume V_0 for which expansions (1.2) are valid is chosen in some vicinity of the point $\mathbf{r}_0$. It must be sufficiently small in comparison with the dimensions of the system in order for all of the smoothly varying functions within the system to be treated as constant. At the same time, its linear dimensions must be sufficiently large in comparison with the wavelength of those perturbations which are intensively attenuated as a result of viscosity. This choice of the volume V_0 is possible if the Reynolds number for the turbulent flow under consideration is large.

2. The equations of motion for the Fourier amplitude and the average field. The functions $\mathbf{u}(\mathbf{r}, t)$ and $p(\mathbf{r}, t)$ are the solutions for the Navier-Stokes equations [2] and for the continuity equation

$$\partial \mathbf{u} / \partial t + (\mathbf{u}\nabla)\mathbf{u} = -\rho^{-1} \operatorname{grad} p + \nu\Delta\mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0. \quad (2.1)$$

Here ρ is the fluid density and ν is the kinematic viscosity.

To find the system of equations which is satisfied by the Fourier amplitudes of expansions (1.2), let us substitute (1.1) into (2.1), and after having multiplied the left- and right-hand members by $\exp(-i\mathbf{q}\mathbf{r}_0)$, we will integrate the resulting equation over the volume V_0. Restricting ourselves only to the highest terms of the expansion in powers of kL, we obtain

$$\frac{d}{dt}\mathbf{u}(\mathbf{q}, t) + (\mathbf{u}(\mathbf{q}, t)\nabla)\mathbf{U} + i\Sigma(\mathbf{q}'\mathbf{u}(\mathbf{k}, t))\mathbf{u}(\mathbf{q}-\mathbf{k}, t) = \quad (2.2)$$
$$= -\rho^{-1}\mathbf{q}' p(\mathbf{q}, t) - \nu(q'^2 - i \operatorname{div} \mathbf{q}')\mathbf{u}(\mathbf{q}, t).$$

The derivatives of the functions $U(\mathbf{r}, t)$ with respect to r in Eq. (2.2) should be taken at the point $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$. The vector $\mathbf{q}' = \mathbf{q}'(\mathbf{q}, \mathbf{r}_0, t)$ resulting from the transition to Lagrange variables is defined as follows: $q_x' = q_x \partial x_0/\partial x + q_y \partial y_0/\partial x + q_z \partial z_0/\partial x$, and, analogously, the components q_y' and q_z'.

It follows from continuity equation (2.1) that

$$\operatorname{div} \mathbf{U} = 0 \qquad (\mathbf{q}' \, \mathbf{u}(\mathbf{q}, t)) = 0.$$

Since the component of the vector $\mathbf{u}(\mathbf{q}, t)$ along $\mathbf{q}'$ is equal to zero, we can assume that

$$\begin{aligned} u_x(\mathbf{q}, t) &= \cos\theta' \cos\varphi' \, a(\mathbf{q}, t) - \sin\varphi' \, b(\mathbf{q}, t), \\ u_y(\mathbf{q}, t) &= \cos\theta' \sin\varphi' \, a(\mathbf{q}, t) + \cos\varphi' \, b(\mathbf{q}, t), \\ u_z(\mathbf{q}, t) &= -\sin\theta' \, a(\mathbf{q}, t). \end{aligned} \quad (2.3)$$

Here θ' and φ' are the angles of the vector $\mathbf{q}'$ in a spherical system of coordinates.

The transition to the equations for $a(\mathbf{q}, t)$ and $b(\mathbf{q}, t)$ can be accomplished if (2.3) is substituted into (2.2) and if we eliminate $p(\mathbf{q}, t)$:

$$da(\mathbf{q}, t)/dt = A_{11} a(\mathbf{q}, t) + A_{12} b(\mathbf{q}, t) -$$
$$- i \sum_{\mathbf{k}_1, \mathbf{k}_2} \delta_{\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2} q' \{[\sin\theta' \cos\theta_1' \cos(\varphi' - \varphi_1') -$$
$$- \cos\theta' \sin\theta_1'] \, a(\mathbf{k}_1, t) + \sin\theta' \sin(\varphi' - \varphi_1') \, b(\mathbf{k}_1, t)\} \times$$
$$\times \{[\cos\theta' \cos\theta_2' \cos(\varphi' - \varphi_2') +$$
$$+ \sin\theta' \sin\theta_2'] \, a(\mathbf{k}_2, t) + \cos\theta' \sin(\varphi' - \varphi_2') \, b(\mathbf{k}_2, t)\},$$
$$db(\mathbf{q}, t)/dt = A_{21} a(\mathbf{q}, t) + A_{22} b(\mathbf{q}, t) -$$
$$- i \sum_{\mathbf{k}_1, \mathbf{k}_2} \delta_{\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2} q' \{[\sin\theta' \cos\theta_1' \cos(\varphi' - \varphi_1') - \cos\theta' \sin\theta_1'] \times$$
$$\times a(\mathbf{k}_1, t) + \sin\theta' \sin(\varphi' - \varphi_1') \, b(\mathbf{k}_1, t)\} \times$$
$$\times \{-\cos\theta_2' \sin(\varphi' - \varphi_2') \, a(\mathbf{k}_2, t) + \cos(\varphi' - \varphi_2') \, b(\mathbf{k}_2, t)\},$$
$$A_{11} = -\nu q'^2 + i\nu \operatorname{div} \mathbf{q}' - \cos^2\theta' \cos^2\varphi' \, \partial U_x/\partial x -$$
$$- \cos^2\theta' \sin\varphi' \cos\varphi' \, \partial U_x/\partial y +$$
$$+ \sin\theta' \cos\theta' \cos\varphi' \, \partial U_x/\partial z -$$
$$- \cos^2\theta' \sin\varphi' \cos\varphi' \, \partial U_y/\partial x - \cos^2\theta' \sin^2\varphi' \, \partial U_y/\partial y +$$
$$+ \sin\theta' \cos\theta' \sin\varphi' \, \partial U_y/\partial z + \sin\theta' \cos\theta' \cos\varphi' \, \partial U_z/\partial x +$$
$$+ \sin\theta' \cos\theta' \sin\varphi' \, \partial U_z/\partial y - \sin^2\theta' \, \partial U_z/\partial z,$$

$$A_{12} = \cos\theta' d\varphi'/dt + \cos\theta' \sin\varphi' \cos\varphi' \,\partial U_x/\partial x -$$
$$- \cos\theta' \cos^2\varphi' \,\partial U_x/\partial y + \cos\theta' \sin^2\varphi' \times$$
$$\times \partial U_y/\partial x - \cos\theta' \sin\varphi' \cos\varphi' \,\partial U_y/\partial y -$$
$$- \sin\theta' \sin\varphi' \,\partial U_z/\partial x + \sin\theta' \cos\varphi' \,\partial U_z/\partial z,$$
$$A_{21} = -\cos\theta' d\varphi'/dt + \cos\theta' \sin\varphi' \cos\varphi' \partial U_x/\partial x +$$
$$+ \cos\theta' \sin^2\varphi' \partial U_x/\partial y - \sin\theta' \sin\varphi' \times$$
$$\times \partial U_x/\partial z - \cos\theta' \cos^2\varphi' \,\partial U_y/\partial x -$$
$$- \cos\theta' \sin\varphi' \cos\varphi' \,\partial U_y/\partial y + \sin\theta' \cos\varphi' \,\partial U_y/\partial z,$$
$$A_{22} = -\nu q'^2 + i\nu \operatorname{div} q' -$$
$$- \sin^2\varphi' \,\partial U_x/\partial x + \sin\varphi' \cos\varphi' \,\partial U_x/\partial y +$$
$$+ \sin\varphi' \cos\varphi' \,\partial U_y/\partial x - \cos^2\varphi' \,\partial U_y/\partial y$$

Here θ_1', φ_1', and θ_2', φ_2' are the angles of the vectors k_1' and k_2'.

System (2.4) permits us to solve the problem of the behavior for perturbations whose wavelength is substantially smaller than the dimensions of the system. If the amplitudes of these perturbations are small (for example, at the initial stage of their development), the quadratic terms with respect to the amplitudes can be neglected in (2.4) and we obtain a simple system of linear equations. In investigating this linearized system, for each field $\mathbf{U}(r_0, t)$ it is not difficult to find that region of wave numbers which belongs to the perturbations increasing with time, nor is it difficult to establish the rate of their growth.

The equation for the average field of velocities $\mathbf{U}(r, t)$ can be obtained if we substitute (1.1) into (2.1), and if we then integrate over the volume V_0

$$\partial U_i / \partial t + U_k \partial U_i / \partial x_k = -\rho^{-1} \partial P / \partial x_i - \partial / \partial x_k T_{ik} \qquad (2.5)$$

where, for example,

$$T_{xx} = \sum_q u_x^* (q, t)\, u_x (q, t) =$$
$$= \sum_q [a^* (q, t)\, a (q, t) \cos^2\theta' \cos^2\varphi' -$$
$$- a^* (q, t)\, b (q, t) \cos\theta' \sin\varphi' \cos\varphi' -$$
$$- b^* (q, t)\, a (q, t) \cos\theta' \sin\varphi' \cos\varphi' + b^* (q, t)\, b (q, t) \sin^2\varphi'],$$
$$T_{xy} = \sum_q u_x^* (q, t)\, u_y (q, t).$$

The system of (2.4) and (2.5) is thus closed.

3. **The equation for the distribution function.** If we know the initial conditions and the function $\mathbf{U}(r, t)$, with system (2.4) we can—in principle—trace the evolution of the perturbations $a(q, t)$ and $b(q, t)$. However, in actual fact, to the extent that we are dealing with perturbations whose wavelengths are substantially smaller than the dimensions of the system, it is at no time possible to know the initial data exactly, and we can speak essentially only of the probability with which certain values can be anticipated for the initial amplitudes $a(q, 0)$ and $b(q, 0)$.

It is therefore necessary to introduce the distribution function $F(a_{q_1}, b_{q_1}; \ldots; a_{q_n}, b_{q_n}; \ldots, t)$ and to examine an entire set of systems differing from each other only in the magnitudes of the amplitudes for the initial perturbations. The distribution function must satisfy the partial differential equation

$$\frac{\partial F}{\partial t} + \sum_q \left[\frac{\partial}{\partial a_q} L(a_q) + \frac{\partial}{\partial b_q} L(b_q)\right] F = 0, \qquad (3.1)$$

whose characteristics are the equations of (2.4). Hence it follows that $L(a_q)$ and $L(b_q)$ represent the right-hand members of the first and second equations in (2.4), respectively.

Consequently, to obtain exhaustive data on the development of perturbations in a turbulent flow, we must find the distribution function which satisfies Eq. (3.1) and the initial condition

$$F(a_{q_1}, b_{q_1}; \ldots; a_{q_n}, b_{q_n}; \ldots; 0) =$$
$$= F_0(a_{q_1}, b_{q_1}; \ldots; a_{q_n}, b_{q_n}; \ldots).$$

4. **The case of spherical-symmetric motion.** For the case of spherical-symmetric motion, when

$$r = r(r_0, t), \qquad \partial r/\partial r_0 = r_0^2/r^2,$$

the coefficients A_{ik} in Eqs. (2.4) have the form

$$A_{11} = -\nu q'^2 + i\nu \operatorname{div} q' - (1 - 3\sin^2\vartheta')\, U / r,$$
$$A_{12} = A_{21} = 0, \qquad A_{22} = -\nu q'^2 + i\nu \operatorname{div} q' - U / r$$

where ϑ' is the angle between the vectors $\mathbf{q}'$ and $\mathbf{r}$; $a(\mathbf{q}, t)$ and $b(\mathbf{q}, t)$ are determined at each point in the coordinate system with the z-axis along $\mathbf{r}$. Since in this case

$$\mathbf{q}' = \mathbf{q}\left[\left(\frac{r}{r_0}\right)^4 \cos^2\vartheta + \left(\frac{r_0}{r}\right)^2 \sin^2\vartheta\right]^{1/2},$$
$$\sin^2\vartheta' = \frac{(r_0/r)^2 \sin^2\vartheta}{(r/r_0)^4 \cos^2\vartheta + (r_0/r)^2 \sin^2\vartheta};$$

where q, ϑ, ψ, are the spherical coordinates, so that $\vartheta \to 0$; if the velocity is directed from the center ($U > 0$, $r/r_0 \to \infty$) and $\vartheta \to \pi/2$; if the velocity is directed to the center $U < 0$, $r/r_0 \to 0$.

The coefficients A_{11} and A_{22} for $U > 0$ are negative if r/r_0 is sufficiently large, independent of the direction of q, and all of the perturbations are therefore attenuated, and the motion is not made turbulent.

With $U < 0$ and small r/r_0, it is only the coefficient A_{11} that is negative for all directions of the vector $\mathbf{q}$. The coefficient A_{22} remains positive in that region of wave numbers in which it is possible to neglect viscosity. Consequently, in the case of convergent spherical-symmetric motion, all amplitudes $a(\mathbf{q}, t)$ are attenuated at some instant of time, whereas the amplitudes $b(\mathbf{q}, t)$, corresponding to a certain region of wave numbers $\mathbf{q}$, increase in size.

The growth of the amplitudes $b(\mathbf{q}, t)$ may restrain nonlinear interaction leading to an exchange of energy between the components of $a(q, t)$ and $b(q, t)$. However, as follows from Eq. (2.4) and as r/r_0 diminishes, this exchange can be neglected, since the terms responsible for this exchange are found to be $\cos\vartheta' \approx (r/r_0)^3$ times smaller than the terms responsible for the exchange of energy between the amplitudes $b(\mathbf{q}, t)$ referred to various $\mathbf{q}$. If, in addition, the viscosity in the system is extremely small, we have

$$\frac{dT}{dt} = -2\frac{U}{r}T, \qquad T = \frac{1}{2}\sum_q |b^2(q, t)|. \qquad (4.1)$$

Finally, omitting the terms which are quadratic with respect to $a(\mathbf{q}, t)$, we write the equation for the velocity $U(r_0, t)$ which follows from (2.5):

$$dU/dt = -\rho^{-1}\, \partial P/\partial r + 2\, T/r. \qquad (4.2)$$

With the aid of system (4.1) and (4.2) we can determine the influence exerted by turbulization on the collapse of the spherical cavity (for the solution of this problem without consideration of trubulization, see, for example, [2]). Let us present the expression for the velocity of the inside boundary in the cavity (which is of interest in this problem)

$$\left(\frac{dR}{dt}\right)^2 = \frac{2}{3}\frac{p_0}{\rho}\left(\frac{R_0^3}{R^3} - 1\right) + \left(\frac{dR_0}{dt}\right)^2 \frac{R_0^3}{R^3} -$$
$$-2\left[\int_{R_0}^{\infty} \frac{T_0 r_0^4 r_0}{(r_0^3 - R_0^3 + R^3)^{2/3}} - \int_{R_0}^{\infty} T_0 r_0^2 dr_0\right] \frac{R_0^3}{R^3}; \qquad (4.3)$$

where R and dR/dt are, respectively, the radius and velocity of the cavity boundary at the instant of time t; R_0 and dR_0/dt denote the same quantities, but at the initial instant of time; T_0 is the initial energy of the small-scale motion per unit mass; P_0 is the pressure at infinity.

It follows from (4.3) that turbulization—if its initial value is sufficiently small—does not alter the law governing the motion of boundary of a spherical cavity in an incompressible fluid for $R \ll R_0$, where $dR/dt \approx 1/R^{3/2}$.

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